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Mean survival probability in a one-dimensional random medium with traps

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Abstract. We study the mean survival probability $\psi(n)$ at time n on a random one-dimensional chain with perfect absorbers at 0 and L. The transition probabilities g_i at the lattice sites i, are independent identically distributed random variables having the distribution $p(g_i) = 1$ for $0 \leqslant g_i \leqslant 1$. We prove the asymptotic inequality, $C_1 \leqslant \frac{\psi(n)n^2}{(\log n)^{L-3}} \leqslant C_2$ where C_1 and C_2 are finite positive constants which depend on the lattice size L, but not on n. We confirm this result by simulations for lattice sizes up to $L = 17$.

Diffusion and transport on disordered systems have been studied extensively since these serve as models of many physical systems such as random field magnets, charged particle diffusion when attached to a Brownian chain etc [1–5]. Some of the commonly studied quantities are the mean and mean-squared distance travelled in time t, the return probability and the mean survival probability in the presence of traps. Quite often one gets anomalous behaviour in such systems. In a commonly studied problem on disordered lattices, one considers a discrete-time random walk on a one-dimensional lattice with only the probabilities, at any site, of taking a step to the right (or left) being random variables. Sinai [6] found that if $\langle \ln p \rangle = \langle \ln q \rangle$ where p and q are, respectively, the probabilities to take a step to the right and left, the mean distance travelled in time t is given by $(\ln t)^2$. For the asymmetric case, $\langle \ln p \rangle \geqslant \langle \ln q \rangle$, Derrida and Pomeau [7] found that if $\langle q/p \rangle \geqslant 1$, the mean distance R travelled in time t varies as t^x where x lies between 0 and 1. Bouchaud *et al* [5] have carried out a detailed analysis of the continuous version of this problem. The mean survival probability in the presence of traps at time t is another quantity of interest which has been widely studied. Here, one may distinguish between various cases. The simplest case is that with uniform transition rates and with two fixed traps a distance l apart. In this case the survival probability $\psi(t)$ goes as $\exp(-Dt/l^2)$ where D, the diffusion coefficient, depends on the transition rates and the lattice spacing. In a second kind of problem, the transition rates are uniform but the distribution of traps is random. In this case, the quantity of interest is the disorder average of the survival probability. Donsker and Varadhan [8] have solved the multi-dimensional problem rigorously and they show that the survival probability $\psi(n)$ for a D-dimensional lattice is given asymptotically by $\ln(\psi(n)) \sim -a(\ln(1/(1-c))^{2/(D+2)} n^{D/(D+2)}$. Here, a is a constant which depends on the lattice and c is the concentration of traps.

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In this paper, we consider a third type of problem. This is the case of disordered transition rates with two fixed traps at lattice points 0 and L, respectively. The probabilities for transition at site i are g_i to the right (from i to $i+1$) and $1-g_i$ to the left (from i to $i-1$). These transition probabilities are themselves identically distributed independent random variables with the probability distribution $p(g_i) = 1$ for $0 \leqslant g_i \leqslant 1$ for $i = 1, 2, \ldots, L-2, L-1$. For the mean survival probability $\psi(n)$ at time n, we analytically prove the following asymptotic inequality (valid for suficiently large n):

$$C_1 \leqslant \psi(n)n^2/(\log n)^{L-3} \leqslant C_2 \tag{1}$$

where C_1 and C_2 are finite positive constants which depend on L but not on n. We also confirmed these results with simulations for L values ranging from 5 to 17. Note that the distribution is symmetric and corresponds to the Sinai case.

The mean survival probability involves a double averaging; one over the random walks and one over different realizations of the lattice. Formally,

$$\psi(n) = \sum_{\{g_i\}} \sum_{\Omega_n} P(\{g_i\}, \Omega_n) X(\Omega_n, n). \tag{2}$$

Here $P(\{g_i\}, \Omega_n)$ is the joint probability of getting a realization with transition probabilities $\{g_i\}$ and an n step random walk Ω_n on this realization. $X(\Omega_n, n)$ is an indicator function which is equal to one if the particle survives up to n in random walk Ω_n, and zero otherwise.

The method used in this paper originates from the work of Toth and Knight (see [9, 10]). The basic idea used is that there is a one-to-one correspondence between the set of all random walks and the sequence of right and left steps at each of the lattice points [9]. Let $\Omega_n = (x_0, x_1, \ldots, x_n)$ be a random walk of n steps where x_j is the position of the random walker at time j. We restrict our walk so that these will range from i_1 to i_2 that is $i_1 \leqslant x_j \leqslant i_2$ for $j = 0, 1, \ldots, n$, and there exist time steps j_1 and $j_2 (1 \leqslant j_1, j_2 \leqslant n)$ such that $x_{j_1} = i_1$ and $x_{j_2} = i_2$. Now, the sequence $(x_0, x_1, \ldots, x_n)$ uniquely specifies the sequence of steps (right or left) at every lattice point $i (i_1 \leqslant i \leqslant i_2)$. Conversely, if one specifies the initial point as well as the total number of right and left steps and the order in which they occur at each of the lattice points $i_1, i_1+1, \ldots, i_2$ one can obtain the unique sequence $\Omega = (x_0, x_1, \ldots, x_n)$. Not all values of r_i and l_i are, however, allowed. They have to satisfy some constraints. For the case $i_1 \leqslant x_0 \leqslant x_n \leqslant i_2$ we have the following constraints:

$$\begin{aligned}
&l_{i_1} = 0 \qquad r_{i_2} = 0 \\
&r_i \geqslant 1 \qquad \text{for} \quad i_2 > i \geqslant x_0 \\
&l_i \geqslant 1 \qquad \text{for} \quad i_1 < i \leqslant i_0 \\
&\sum_{i=i_1}^{i_2} (r_i + l_i) = n \\
&l_i = r_{i-1} \qquad \text{for} \quad i_1 < i \leqslant x_0 \qquad \text{and} \qquad x_n < i \leqslant i_2 \\
&l_i = r_{i-1} - 1 \qquad \text{for} \quad x_0 < i \leqslant x_n.
\end{aligned} \tag{3}$$

To keep things simple, we consider the return probability (i.e. the probability that a particle starting from lattice site i_0 is found at i_0 after n steps). We further take n to be even $(= 2m)$, since the particle can return to the place it started from only after an even number of steps. The probability $P(\Omega_n | \{g_i\})$ for obtaining a random walk Ω_n given a realization with the set of transition probabilities $\{g_i\}$ can be written as a product

$$P(\Omega_n | \{g_i\}) = \prod_{i=i_1}^{i_2} Q_i. \tag{4}$$

Q_i is the probability for the steps starting from lattice site i in the random walk Ω_n in a realization with the transition probabilities which are $g_{i_1}, g_{i_1+1}, \ldots, g_{i_2}$. If the particle took r_i steps to the right and l_i to the left at site i, then

$$Q_i = g_i^{r_i}(1-g_i)^{l_i}. \tag{5}$$

Note that this probability is independent of the order in which the right and left steps were taken. Since the g_i are independent random variables, the Q_i can be independently averaged over all realizations:

$$\bar{Q}_i = \int p(g_i)\,\mathrm{d}g_i\,Q_i. \tag{6}$$

In our case, $p(g_i) = 1$ for $0 < g_i < 1$. Therefore, we get

$$\bar{Q}_i = \frac{r_i!l_i!}{(r_i+l_i+1)!}. \tag{7}$$

The total probability $R_{i_1,i_2}(\Omega_n, n)$ for the walk Ω_n having a range from i_1 to i_2 averaged over all realizations is given by

$$R_{i_1,i_2}(\Omega_n, n) = \prod_{i=i_1}^{i_2} \bar{Q}_i. \tag{8}$$

The total return probability $R(n)$ of a particle starting from i_0 is given by

$$R(n) = \sum_{i_1=1}^{i_0} \sum_{i_2=i_0}^{L-1} \sum_{\Omega_n} R_{i_1,i_2}(\Omega_n, n) \tag{9}$$

where the sum over Ω_n runs over all random walks which start from i_0 and return to i_0 at the nth step. Because (i) there is a one-to-one correspondence between the n step random walks and the sequence of steps at each of the lattice points, and (ii) the probability Q_i depends only on the number of right and left steps at i and not on their order, the sum over Ω_n may be replaced by a sum over all possible values of $r_{i_1}, r_{i_1+1}, \ldots, r_{i_2-1}$, which satisfy the constraints of equation (3):

$$R_{1,L-1}(n) = \sum_{r_1\geqslant 1} \cdots \sum_{r_{L-2}\geqslant 1} \prod_{i=1}^{L-1} N_i\,\bar{Q}_i\,\delta\left[\sum_{i=1}^{L-1}(r_i+l_i) - n\right]. \tag{10}$$

Here N_i is the number of ways the r_i steps to the right and l_i steps to the left can be arranged. If i is to the right of i_0, then the last step taken at i has to be to the left and therefore,

$$N_i = \frac{(r_i+l_i-1)!}{(r_i)!(l_i-1)!}. \tag{11}$$

Similar expressions can be written for sites to the left of i_0 as well as i_0 itself:

$$\begin{aligned} N_i\,\bar{Q}_i &= \frac{(r_i+l_i-1)}{r_i!(l_i-1)!}\,\frac{r_i!l_i!}{(r_i+l_i+1)!} \\ &= \frac{l_i}{(r_i+l_i)(r_i+l_i+1)}. \end{aligned} \tag{12}$$

Further, for the return probability we have $l_i = r_{i-1}$ for $i = 2, 3, \ldots, L-1$. Using equation (12) in (10), we get, for $2 \leqslant i_0 \leqslant L-2$

$$R_{1,L-1}(n) = \sum_{r_1} \cdots \sum_{r_{L-2}} \frac{1}{(r_1+1)} \prod_{i=2}^{i_0-1} \frac{r_i}{(r_{i-1}+r_i)(r_{i-1}+r_i+1)}$$

$$\times \frac{1}{(r_{i_0-1} + r_{i_0} + 1)} \prod_{i=i_0+1}^{L-2} \frac{r_{i-1}}{(r_{i-1} + r_i)(r_{i-1} + r_i + 1)}$$
$$\times \frac{1}{(r_{L-2} + 1)} \delta\left[\sum_{i=1}^{L-2} r_i - m\right] \tag{13}$$

where, $m = n/2$.

The probability of finding the particle at site j given that it started from i_0 may be written in a similar manner. We now obtain upper and lower bounds for the return probability.

Upper bound. We first obtain an upper bound for the term $R_{1,L-1}(n)$. Clearly, since $r_{i-1} + r_i \geqslant r_i$, we obtain the following inequality from equation (13):

$$R_{1,L-1}(n) \leqslant \sum_{r_1=1}^{m} \cdots \sum_{r_{L-2}=1}^{m} \frac{1}{(r_1 + 1)} \frac{1}{(r_1 + r_2 + 1)}$$
$$\times \frac{1}{(r_2 + r_3 + 1)} \cdots \frac{1}{(r_{L-3} + r_{L-2} + 1)} \frac{1}{(r_{L-2} + 1)} \delta\left[\sum_{i=1}^{L-1} r_i - m\right]$$
$$:= R^1_{L-1}(n). \tag{14}$$

Since we are interested only in asymptotics, we can convert the sums into integrals and the Kronecker delta function to a Dirac delta function. Using the delta function to carry out the integral over r_{L-2}, we get

$$R^1_{L-1}(n) = \int_1^{u_1} \int_1^{u_2} \cdots \int_1^{u_{L-3}} \mathrm{d}r_1 \ldots \mathrm{d}r_{L-3}$$
$$\times \frac{1}{(r_1 + 1)} \prod_{i=2}^{L-4} \frac{1}{r_{i-1} + r_i + 1} \frac{1}{(r_{L-4} + r_{L-3} + 1)}$$
$$\times \frac{1}{(u_{L-3} + 2)} \frac{1}{(u_{L-2} + 1)}. \tag{15}$$

Here u_i is defined by

$$u_i = m - \sum_{j=1}^{i-1} r_j - L + 2 + i \qquad \text{for} \quad i > 1$$
$$u_1 = m - L + 1. \tag{16}$$

Clearly,

$$u_i = u_{i+1} + r_i - 1. \tag{17}$$

The integral over r_{L-3} can, therefore, be written as

$$\int_1^{u_{L-3}} \frac{\mathrm{d}r_{L-3}}{(r_{L-4} + r_{L-3} + 1)(u_{L-3} - r_{L-3} + 2)}. \tag{18}$$

After integration, we get

$$\frac{1}{(u_{L-3} + r_{L-4} + 3)} \left[\ln \frac{r_{L-4} + u_{L-3} + 1}{r_{L-4} + 2} + \ln \frac{u_{L-3} + 1}{2}\right]. \tag{19}$$

The quantity inside the square parenthesis is $\leqslant 2\log(m)$. Using this as well as equation (17), and substituting the value of the integral over r_{L-3} in equation (15), we get

$$R^1_{L-1}(n) \leqslant 2\log m \int_1^{u_1} \int_1^{u_2} \cdots \int_1^{u_{L-4}} \mathrm{d}r_1 \ldots \mathrm{d}r_{L-4}$$

$$\times \frac{1}{(r_1+1)} \prod_{i=2}^{L-5} \frac{1}{r_{i-1}+r_i+1} \frac{1}{(r_{L-5}+r_{L-4}+1)}$$
$$\times \frac{1}{(u_{L-4}+4)} \frac{1}{(u_{L-3}+2)}. \tag{20}$$

It is easy to see that the following inequality is, therefore, satisfied.

$$R^1_{L-1}(n) \leqslant R^1_{L-2}(n) \cdot 2\log m. \tag{21}$$

It is easy to show that $R_4(n) \leqslant C(\log m)/m^2$. Using this inequality with equation (21), we get

$$R^1_{L-1}(n) \leqslant C(\log m)^{(L-3)}/m^2 \qquad \text{for} \quad L \geqslant 4. \tag{22}$$

Now $R(n)$ is the return probability to site i_0. It is given by

$$R(n) = \sum_{i=1}^{i_0} \sum_{j=i_0}^{L-1} R_{ij}(n). \tag{23}$$

There are less than L^2 terms in the sum and each is less than equal to $R_{1,L-1}(n)$. Therefore, for large enough n, we have for $L \geqslant 4$

$$\frac{R(n)m^2}{(\log m)^{L-3}} \leqslant C. \tag{24}$$

Lower bound. We first note that the term T containing r_{i_0} in equation (14) is

$$T = \frac{r_{i_0}}{(r_{i_0-1}+r_{i_0}+1)(r_{i_0}+r_{i_0+1})(r_{i_0}+r_{i_0+1}+1)}. \tag{25}$$

Now, we restrict the upper limit of integration for the variables $r_1, r_2, \ldots, r_{i_0-1}, r_{i_0+1}, \ldots, r_{L-2}$ to m/L. This will lead to an underestimate. Further, to satisfy the delta function constraint, r_{i_0} has to be greater than m/L and, therefore, greater than r_{i_0-1} as well as r_{i_0+1}. Using this, we can remove the integral over r_{i_0} as well as the delta function in equation (14) and write the inequality

$$R_{1,L-1}(n) \geqslant \frac{1}{m^2} \int_1^{m/L} \cdots \int_1^{m/L} \mathrm{d}r_1 \ldots \mathrm{d}r_{i_0-1}\, \mathrm{d}r_{i_0+1} \ldots \mathrm{d}r_{L-2} \frac{1}{(r_1+1)}$$
$$\times \prod_{i=2}^{i_0-1} \frac{r_i}{(r_{i-1}+r_i)(r_{i-1}+r_i+1)}$$
$$\times \prod_{i=i_0+2}^{L-2} \frac{r_{i-1}}{(r_{i-1}+r_i)(r_{i-1}+r_i+1)} \frac{1}{(r_{L-2}+1)}. \tag{26}$$

This allows us to write $R_{1,L-1}(n)$ as

$$R_{1,L-1}(n) \geqslant \frac{1}{8m^2} T_1 T_2 \tag{27}$$

where

$$\begin{aligned} T_1 &= \int_1^{m/L} \cdots \int_1^{m/L} \mathrm{d}r_1 \ldots \mathrm{d}r_{i_0-1} \frac{1}{(r_1+1)} \prod_{i=2}^{i_0-1} \frac{r_i}{(r_{i-1}+r_i)(r_{i-1}+r_i+1)} \\ T_2 &= \int_1^{m/L} \cdots \int_1^{m/L} \mathrm{d}r_{i_0+1} \ldots \mathrm{d}r_{L-2} \prod_{i=i_0+2}^{L-2} \frac{r_{i-1}}{(r_{i-1}+r_i)(r_{i-1}+r_i+1)} \frac{1}{(r_{L-2}+1)}. \end{aligned} \tag{28}$$

We further restrict the range of integration as follows—in T_1 the lower limit for r_i is raised to r_{i-1} (for r_1, it continues to be one). This allows us to write

$$\frac{r_i}{(r_{i-1}+r_i)(r_{i-1}+r_i+1)} \geqslant \frac{1}{4(r_i+1)}. \tag{29}$$

Using equation (29) in (28), we have

$$T_1 \geqslant \int_1^{m/L} \int_{r_1}^{m/L} \cdots \int_{r_{i_0-2}}^{m/L} \mathrm{d}r_1 \ldots \mathrm{d}r_{i_0-1} \prod_{i=1}^{i_0-1} \frac{1}{4(r_i+1)}. \tag{30}$$

This leads us to the asymptotic inequality (we drop the terms of the type $\log L$ in comparison to $\log m$)

$$T_1 \geqslant \frac{\log^{i_0-1} m}{4^{i_0-2}(i_0-1)!}. \tag{31}$$

We can obtain a similar inequality for T_2

$$T_2 \geqslant \frac{\log^{L-2-i_0} m}{4^{L-3-i_0}(L-3-i_0)!}. \tag{32}$$

Combining equations (27), (31) and (32), we have, asymptotically,

$$R_{1,L-1}(n) \frac{m^2}{\log^{L-3} m} \geqslant C(L). \tag{33}$$

Clearly the total return probability $R(n) \geqslant R_{1,L-1}(n)$. Combining equations (24) and (33), we get for sufficiently large n

$$C_1 \leqslant \frac{R(n) n^2}{(\log n)^{L-3}} \leqslant C_2. \tag{34}$$

Since the mean survival probability $\psi(n)$ consists of $L-1$ terms of the same order as $R(n)$, the result of equation (34) holds for $\psi(n)$ also.

Simulations. There are two averages involved in this problem—one over all realizations and the second over random walks in a realization. Only the first part was done using simulations. Having generated a realization, the average survival probability at any time n was obtained by setting up the $(L-1) \times (L-1)$ transition probability matrix A and raising it to the nth power. To reduce computations, n was restricted to values which were powers of two. A number of realizations were generated and the average over all these obtained.

A source of difficulty in this problem is the fact that the major contribution comes from only a very small fraction of the total number of realizations. To improve the statistics, we used a biased sampling procedure for the realizations. The biasing scheme was as follows:

$$\begin{aligned} &q(g_i) = m g_i^{m-1} && 1 \leqslant i < i_0 \\ &q(h_i) = m h_i^{m-1} && L-1 \geqslant i > i_0 \\ &q(g_{i_0}) = 1 \end{aligned} \tag{35}$$

where i_0 is the starting point of the random walker, h_i is the transition probability to the left (from i to $i-1$) and q was the biased distribution. To get an unbiased estimate, the realizations were given a weight w given by

$$w = \prod_{i=1}^{i_0-1} \frac{1}{g_i} \prod_{i=i_0+1}^{L-1} \frac{1}{q(h_i)}. \tag{36}$$

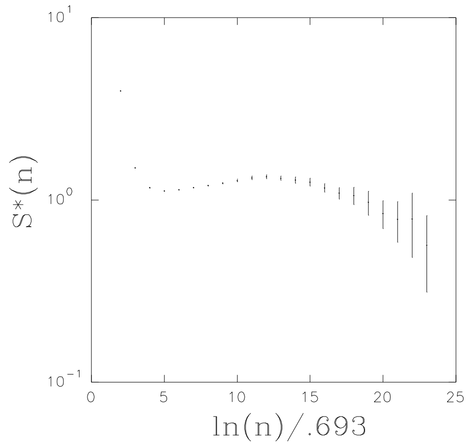

Figure 1. Plot of $\ln(S^*(n))$ versus $\ln(n)$ for a lattice of size 5.

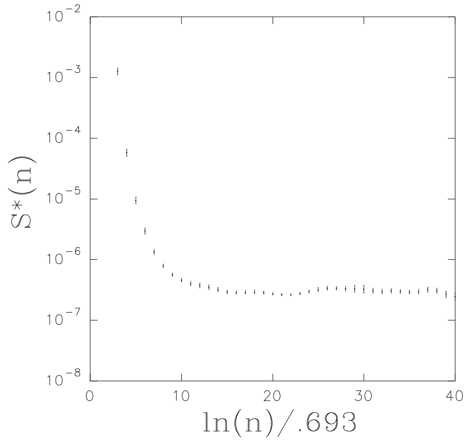

Figure 2. Plot of $\ln(S^*(n))$ versus $\ln(n)$ for a lattice of size 13.

The values used in the simulations for m varied from 2 to 6. Two criteria were used to choose the m value for any L: (i) the standard deviation should be minimal; and (ii) the mean should be reasonably constant with changes in m around that value. The reason for using this particular biasing scheme was to force the particle towards the centre. This artificially increases the probability of those realizations which contribute to the quantity of interest. The number of realizations generated varied from 10^7 to 10^8. The standard deviation over most of the range varied from 5 to 10%.

Discussion. The results of the simulations for $L = 5$ and for $L = 13$ are given in figures 1 and 2. The quantity plotted is $S^*(n) = \psi(n)n^2/(\ln n)^{L-3}$. We see that this quantity is reasonably constant over a range of n values varying by a factor of a few thousand which confirms the conclusions of equation (34). Thus, unlike the case of the regular lattice where there is a exponential decay, we have a power law decay. This is consistent with the highly sub-diffusive transport in a Sinai lattice—the mean-squared distance travelled in time n goes as $(\log n)^4$ rather than $n^{0.5}$. The basic reason for the behaviour in the diffusion problem as well as the trapping problem on the Sinai lattice is that fluctuations produce deep potential wells in which the particle gets confined for long periods of time. There is, however, a major difference between the two cases. In the diffusion problem, (see [4, 5]) in almost every realization of the lattice of length x there is a region with a barrier height of order $\sqrt{\sigma x}$, where σ is a quantity which depends on the distribution of the transition probabilities. By the Arrhenius formula, the time t taken to get out of this barrier is of order $\exp(\sqrt{\sigma x})$. Solving the resulting equation for x yields the result that the mean squared distance travelled is $\propto (\ln(t))^4$. In the trapping problem, on the other hand, the contribution to the mean survival probability comes only from a very small fraction of the realizations. These realizations are such that the survival probability is almost one in them. Even though the total probability of occurrence of these realizations is quite small, they provide the dominant contribution to the result because of the high survival probabilities.

In this paper we have derived the power law result for a specific probability distribution. It is expected that a similar result will hold for all Sinai lattices whatever the distribution. However, the actual exponent is likely to depend on the type of distribution of the transition probabilities.

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